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On continuous solutions of a generalized Cauchy–Riemann system with more than one singularity

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Abstract

We study the solvability of the Riemann–Hilbert problem for a generalized Cauchy–Riemann system with several singularities and reveal several new phenomenon. For the number of continuous solutions we shall show that it depends not only on the index but also on the location and type of the singularities; moreover, it does not depend continuously on the coefficients of the equation.

Keywords: Boundary value problem; Singular coefficient; Generalized analytic function; A priori estimate; Continuous solution

1. Introduction

Let Ω be the unit disk $\{z\colon\ |z|<1\}$ in the complex plane $\mathbb{C}$. In this paper we consider a generalized Cauchy–Riemann system

$$w_{\bar z} = \frac{Q(\bar z)}{P(\bar z)}w + aw + b\bar w, \tag{1}$$

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where $Q(z)$ and $P(z)$ are polynomials of the complex variable z, and the polynomial P has only simple roots within the closed disk $\bar{\Omega}$.

We shall consider the following Riemann–Hilbert problem for the system of Eq. (1):

Problem RH. Find a regular solution $w(z)$ of system (1) in Ω, continuous in Ω and $w_{\bar{z}}, w_z \in L^p(\Omega)$ for some $p > 2$, satisfying the boundary value condition

$$\mathrm{Re}[\overline{G(z)}w(z)] = g(z), \quad z \in \partial\Omega \tag{2}$$

on the boundary $\partial\Omega$ of the domain Ω, where G and g are given functions on $\partial\Omega$.

For the first-order elliptic system of equations

$$u_{\bar{z}} = A(z)u + B(z)\bar{u}$$

with regular coefficients $A, B \in L^p(\Omega)(p > 2)$ when Ω is a bounded domain in the complex plane $\mathbb{C}$ or $A, B \in L^{p,2}(\mathbb{C})$ function theory, boundary value problems and their generalizations were investigated extensively over the past years, see, e.g. [1–8,17,19]. This regularity allows to use a similarity principle for solutions of regular coefficient systems.

In shell theory, in connection with infinitesimal bends of first or higher order surfaces of positive curvature with some flat point and in connection with strain of thin momentless elastic shells, elliptic systems of equations with singular coefficients occur [18]:

$$w_{\bar{z}} + \frac{A(z)}{\bar{z}} w + \frac{B(z)}{\bar{z}} \bar{w} = 0.$$

Now the coefficients do not belong to the regularity class L^p or $L^{p,2}$, $p > 2$. In [16] a model equation

$$w_{\bar{z}} = \frac{b}{\bar{z}} \bar{w},$$

was investigated, where b is a complex constant. In [10] the equation

$$w_{\bar{z}} + \frac{a(z)}{|z|} w + \frac{b(z)}{|z|} \bar{w} = 0 \tag{3}$$

was studied. It was proved that there exist solutions admitting singularities of order $v > 0$ at the point $z = 0$. Eq. (3) when $a(z) = 0$ was perturbed as

$$w_{\bar{z}} + \frac{b(0)}{\bar{z}} \bar{w} + \frac{b(z) - b(0)}{\bar{z}} \bar{w} = 0.$$

Under the assumption that $(b(z) - b(0))/\bar{z}$ is sufficiently small, the existence of continuous solutions was studied in [16].

Observing that Eq. (3) is singular only at the origin and elsewhere its coefficients are regular, in [14] both cases were treated separately first and then these two solutions were glued together.

In [6,7] we found through the model equation

$$w_{\bar{z}} = \frac{\lambda}{\bar{z}} w + aw + b\bar{w}, \quad a, b \in L^p(\Omega), \quad p > 2,$$

that the number of continuous solutions depends on size and sign of the constant λ. This observation was implicitly supported by the results in [13], where the model equation

$$\psi_{\bar{z}} + \frac{a}{2\bar{z}}\psi + \frac{b}{2\bar{z}}\bar{\psi} = 0$$

was studied, where a and b are complex constants.

Throughout this paper we assume that

(1) Let $P(z) = \prod_{l=1}^{m}(z - \overline{z_l})$, $|z_l| \leqslant 1$ $(l = 1, 2, \ldots, m)$ and we assume that $\{z_l\}$ are different points in $\bar{\Omega}$ and that the rational polynomial $Q(z)/P(z)$ can be decomposed as

$$\frac{Q(z)}{P(z)} = \sum_{l=1}^{m} \frac{a_l}{z - \overline{z_l}},$$

where $a_l = Q(\overline{z_l})/P'(\overline{z_l}) \neq 0$ $(l = 1, 2, \ldots, m)$.

(2) $a, b \in L^p(\Omega)(p > 2)$.

(3) G, $g \in C^\alpha(\partial\Omega)(1/2 < \alpha < 1)$, $G \neq 0$ and $\kappa = \mathrm{Ind}_{\partial\Omega}\, G = \frac{1}{2\pi}\Delta_{\partial\Omega} \arg G$ is an integer.

We study the model Eq. (1) from different point of views. The results of this paper will reveal flaws of [14,15]. Moreover, we shall show that the number of continuous solutions of Problem RH depends on the constants a_l and the location of the zeros of the polynomial $P(z)$. These results suggest essential difficulties to obtain a general theory for singular complex systems of equations.

The organization of this paper is as follows: in Section 2, we study the case where all the zeros of the polynomial $P(z)$ are within Ω. The results are applied to several models existing in literature, which show necessity for us to study the model (1). As preparations, a non-normal Riemann–Hilbert problem for generalized analytic functions will be investigated in Section 3. In contrast to the normal Riemann–Hilbert problem a necessary condition for the solvability of the non-normal Riemann–Hilbert problem will be derived. We also propose a new modified boundary condition to ensure existence and uniqueness of solution. Finally, in Section 4 we assume that all the zeros of the polynomial $P(z)$ are on the boundary $\partial\Omega$. For the solvability of Problem RH we derive the necessary and sufficient conditions

$$g(z_l) = 0 \quad (l = 1, 2, \ldots, m),$$

see (54) below, which are not needed when singularities occur only within Ω. Our main result is summarized in Theorem 6.

2. Problem RH when $|z_l|<1$ $(l=1,2,\dots,m)$

In this section we assume that all the zeros of the polynomial $P(z)$ are within Ω, that is we have $|z_l|<1$, $l=1,2,\dots,m$.

We denote by n_l the integral part of the real part of a_l,

$$\operatorname{Re}(a_l)=n_l+\lambda_l,\quad 0\leqslant\lambda_l<1\quad (l=1,2,\dots,m).$$

For $l=1,2,\dots,m$, we set

$$w_l(z)=\begin{cases}\dfrac{(\bar z-\overline{z_l})^{n_l}e^{2i\,\operatorname{Im}\,a_l\,\ln|z-z_l|}}{(z-z_l)^{n_l-1}}, & \lambda_l=0,\\[2ex] \dfrac{(\bar z-\overline{z_l})^{n_l}|z-z_l|^{2\lambda_l}e^{2i\,\operatorname{Im}\,a_l\,\ln|z-z_l|}}{(z-z_l)^{n_l}}, & 0<\lambda_l\leqslant\dfrac{1}{2},\\[2ex] \dfrac{(\bar z-\overline{z_l})^{n_l}|z-z_l|^{2\lambda_l}e^{2i\,\operatorname{Im}\,a_l\,\ln|z-z_l|}}{(z-z_l)^{n_l+1}}, & \dfrac{1}{2}<\lambda_l<1.\end{cases}$$

For the regularity of $w_l(z)$ we have the following results.

Lemma 1. *Let the function $w_l(z)$ be defined as above. Then w_l is continuous on $\bar\Omega$. Moreover*

(1) $w_l\in W^1_p(\Omega)$, $p>2$, *if* $\lambda_l=0$,
(2) $w_l\in W^1_p(\Omega)$, $p<\frac{2}{1-2\lambda_l}$ *if* $0<\lambda_l\leqslant\frac{1}{2}$,
(3) $w_l\in W^1_p(\Omega)$, $p<\frac{1}{1-\lambda_l}$, *if* $\frac{1}{2}<\lambda_l<1$.

Proof. By direct calculations. □

For $l=1,2,\dots,m$ we denote

$$\sigma_l=\begin{cases}2n_l-1, & \lambda_l=0,\\ 2n_l, & 0<\lambda_l\leqslant\frac{1}{2},\\ 2n_l+1, & \frac{1}{2}<\lambda_l<1.\end{cases}$$

For the solvability of Problem RH we have

Theorem 1. *Let $\kappa_0=\kappa+\sum_{l=1}^m\sigma_l$. Then we have*

(1) *when $\kappa_0\geqslant0$, Problem RH is always solvable and the corresponding homogeneous problem has exactly $2\kappa_0+1$ linearly independent solutions over the field of real numbers*;

(2) *when* $\kappa_0 < 0$, *Problem RH is solvable if and only if the function* g *satisfies* $2|\kappa_0| - 1$ *solvability conditions.*

Proof. Let $w_0(z) = \prod_{l=1}^m w_l(z)$. Then by virtue of Lemma 1 and the chain rule of differentiation we have $w_0 \in W^1_{p_1}(\Omega)$ for some $p_1 > 2$, which depends on λ_l ($l = 1, 2, \ldots, m$) as shown in Lemma 1.

By means of the transform

$$w(z) = w_0(z)\varphi(z), \tag{4}$$

Problem RH becomes

$$\begin{cases} \varphi_{\bar{z}} = a\varphi + b_0\bar{\varphi} & \text{in } \Omega, \\ \mathrm{Re}[\overline{G_0(z)}\varphi(z)] = g(z) & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $b_0(z) = b(z)\frac{\overline{w_0(z)}}{w_0(z)}$, $G_0(z) = \overline{w_0(z)}G(z)$.

We seek solutions of (5) such that $\varphi \in C^0(\bar{\Omega} \setminus \bigcup_{l=1}^m \{z_l\}) \cap W^1_{p_2}(\Omega_\varepsilon)$ for some $2 < p_2 \leqslant p_1$ and $\varphi(z) = O(|z - z_l|^{-\alpha_l})$ ($l = 1, 2, \ldots, m$), where $\Omega_\varepsilon = \Omega \setminus \bigcup_{l=1}^m \{|z - z_l| \leqslant \varepsilon\}$, and $\alpha_l < 1$ if $\lambda_l = 0$, $\alpha_l < 2\lambda_l$ if $0 < \lambda_l \leqslant 1/2$, $\alpha_l < 2\lambda_l - 1$ if $1/2 < \lambda_l < 1$.

Noticing that when $w \in C^0(\bar{\Omega})$, $w_{\bar{z}} \in L^{p_2}(\Omega)$ for $p_2 > 2$, from Eq. (1) we must have $w(z_l) = 0$ ($l = 1, 2, \ldots, m$), then it is easily verified that Problem RH is equivalent to problem (5).

By the similarity principle [17], solutions of problem (5) can be expressed as

$$\varphi(z) = \Phi(z)e^{\omega(z)},$$

where $\omega(z)$ is Hölder continuous on $\bar{\Omega}$ and $\Phi(z)$ is analytic in Ω except at z_l, $l = 1, 2, \ldots, m$. At each z_l $\Phi(z)$ has a singularity of order α_l, hence $\Phi(z)$ and consequently $\varphi(z)$, has only a removable singularity at the interior point z_l of Ω, which shows that solutions of (5) can be sought in $W^1_{p_2}(\Omega)$.

Since $b_0(z) \in L^p(\Omega) \subset L^{p_2}(\Omega)$ and $G_0 \in C^\alpha(\partial\Omega)$, problem (5) is a regular problem and its index is

$$\kappa_0 = \mathrm{Ind}_{\partial\Omega}\, G_0 = \kappa + \sum_{l=1}^m \sigma_l.$$

From Vekua's theory, problem (5) is always solvable when $\kappa_0 \geqslant 0$ and the corresponding homogeneous problem has exactly $2\kappa_0 + 1$ linearly independent solutions over the field of real numbers. Moreover, when $\kappa_0 < 0$ problem (5) is solvable if and only if the function g satisfies $2|\kappa_0| - 1$ solvability conditions.

Hence the function $w(z)$ defined by (4) solves Problem RH. The proof of Theorem 1 is completed. □

Remark 1. Let $w_{\lambda_1,\lambda_2,\ldots,\lambda_m}$ denote the solution to Problem RH. From Theorem 1 it follows that the number of solutions does not depend continuously on $\{\lambda_1,\lambda_2,\ldots,\lambda_m\}$ when $\lambda_l \to 0^+, \frac{1}{2}^+$. This shows a difference between the singular case and the regular case, cf. [17, Chapter 3, Section 12]. We point out that even if one is only interested in solutions with prescribed singularity this phenomenon for the number of singular solutions occur too.

Remark 2. In [16, p. 72], the following problem:

$$\begin{cases} U_{\bar{z}} + \dfrac{b(0)}{2\bar{z}} U = 0, & |z|<R \quad (4.3) \\ \mathrm{Re}(ie^{i\varphi}U) = 0, & |z| = R \quad (4.4) \end{cases}$$

was introduced as a conjugate problem. The solutions were sought in the class of functions continuous outside $z=0$ and admitting no more than a first order singularity at $z=0$. The author found that (4.3), (4.4) has only the non-trivial solution $U(z) = |z|^{-b(0)}/z$.

In fact, the number of solutions depends on the constant $b(0)$. In Theorem 1 we are dealing with continuous solutions. It can be modified to seek solutions in the above sense also. For example, when $-2<b(0)<-1$, problem (4.3), (4.4) has three linearly independent solutions

$$i|z|^{-b(0)}(1-z^{-2}), \quad z^{-1}|z|^{-b(0)}, \quad |z|^{-b(0)}(1+z^{-2}),$$

having at most a first-order singularity at $z=0$.

Remark 3. In [15] the following boundary value problem was investigated:

$$\begin{cases} w_{\bar{z}} + A(z)w + B(z)\bar{w} = F, & |z|<1, \\ \mathrm{Re}[z^{-\kappa}w(z)] = g(z), & |z|=1, \end{cases} \tag{6}$$

where $A(z) = A_0(z)/\prod_{j=1}^m |z-z_j|$, $B(z) = B_0(z)/\prod_{j=1}^m |z-z_j|$, A_0, B_0, $F \in L^q(\Omega)$ $(q>2)$. Under the assumption that the norms of $A_j(z) = |z-z_j|A(z)$ and $B_j(z) = |z-z_j|B(z)$ are sufficiently small ([15, p. 105] condition (1.2)) and that the integral equation ([15, p. 114], Eq. (4.7))

$$w_0(z) + P_3 w_0(z) = 0$$

has only the trivial solution, Theorem 3 of [15] says that the homogeneous problem has $2\kappa - 2m + 1$ linearly independent solutions over the field of real numbers when $m \leqslant \kappa$.

In [14], problem (6) was studied when $m=1$. The results in [15] are generalizations of [14].

Let

$$A_0(z) = \frac{Q(z)|P(z)|}{P(z)}, \quad B_0(z) = 0.$$

Since $Q(z)$ can be written as

$$Q(z) = \sum_{l=1}^{m} a_l \left[\prod_{\substack{k=1 \\ k \neq l}}^{m} (z - z_k) \right],$$

we choose $\varepsilon > 0$ such that $0 < a_l < \varepsilon \ (l = 1, 2, \ldots, m)$ and A_0 satisfies condition (1.2) of [15]. Then $n_l = 0, 0 < \lambda_1 < \varepsilon\left(<\frac{1}{2}\right)$, and $\sigma_l = 0 \ (l = 1, 2, \ldots, m)$. By virtue of Theorem 1 $\kappa_0 = \kappa$. Hence, the homogeneous problem (6) has $2\kappa + 1$ linearly independent solutions.

However, if we let $-\varepsilon < a_l < 0 \ (l = 1, 2, \ldots, m)$ then $n_l = -1, \frac{1}{2} < \lambda_l$ and $\sigma_l = -1$. By virtue of Theorem 1 we have $\kappa_0 = \kappa - m$ and the corresponding homogeneous problem (6) has $2\kappa - 2m + 1$ linearly independent solutions.

This shows that it does not suffice to consider modules of its coefficients when studying singular equations.

3. An auxiliary problem

In this section we investigate the solvability of the non-normal type Riemann–Hilbert problem

$$\begin{cases} u_{\bar{z}} = Au + B\bar{u} + C & \text{in } \Omega, \\ \operatorname{Re}[\overline{G(z)}\pi(z)u(z)] = g(z) & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $A, B, C \in L^p(\Omega)(p > 2)$, $\pi(z) = \prod_{l=1}^{m}(z - z_l)$, $z_l \in \partial\Omega \ (l = 1, 2, \ldots, m)$. In [9], Sections 11.7–11.9 a general form of this problem is discussed. The particular form of the coefficient in the boundary condition (7) allows us to prove very precise results.

Problem A. Find a solution $u \in C^0(\bar{\Omega} \backslash \bigcup_{l=1}^{m} \{z_l\}) \cap W_p^1(\Omega_\varepsilon)(p > 2)$ of (7) such that $u(z) = O(|z - z_l|^{-\beta_l}) \ (\beta_l < 1, l = 1, 2, \ldots, m)$, where $\Omega_\varepsilon = \Omega \backslash \bigcup_{l=1}^{m} \{|z - z_l| \leqslant \varepsilon\}$.

Remark 4. Since the boundary function $G(z)\overline{\pi(z)}$ in (7) has zeros on $\partial\Omega$, this kind of boundary condition is often called 'non-normal' condition.

We first make some simplifications. Let $s(z)$ be the unique solution of the problem

$$\begin{cases} s_{\bar{z}} = 0 & \text{in } \Omega, \\ \operatorname{Re} s(z) = \arg G(z) - \kappa \arg z & \text{on } \partial\Omega, \\ \operatorname{Im} s(0) = 1. \end{cases}$$

By the transform $u_0(z) = e^{-is(z)}u(z)$ problem (7) becomes

$$\begin{cases} u_{0\bar{z}} = Au_0 + B_0\overline{u_0} + C & \text{in } \Omega, \\ \operatorname{Re}[\bar{z}^{\kappa}\pi(z)u_0(z)] = \tilde{g} & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where $B_0 = \exp\{-2i \operatorname{Re} s\}B$, $\tilde{g} = \exp\{\operatorname{Im} s(z)\}/|G(z)|$.

We therefore study the simplified problem (8) instead of (7). In Section 3.1 we deal with Problem A for analytic functions. To assure uniqueness of solutions we propose a modified problem in Section 3.2. In Section 3.3 we derive an a priori estimate for the solution of problem (8). In Section 3.4 we study solvability of Problem A.

3.1. *Problem A for analytic functions*

We now consider problem (8) for analytic functions. That is we assume that

$$A = B_0 = C = 0, \tag{9}$$

in this section.

Lemma 2. *Let condition* (9) *hold. Then for the homogeneous problem*

$$\begin{cases} u_{0\bar{z}} = 0 & \textit{in } \Omega, \\ \operatorname{Re}[\bar{z}^{\kappa}\pi(z)u_0(z)] = 0 & \textit{on } \partial\Omega, \end{cases} \tag{10}$$

the following holds:

(1) *If* $2\kappa - m \geqslant 0$, *it has exactly* $2\kappa - m + 1$ *linearly independent solutions over the field of real numbers*

$$u_0(z) = \sum_{k=0}^{2\kappa - m} r_k z^k,$$

where the constants r_k *satisfy* $r_k = (-1)^{m-1}(\prod_{l=1}^{m} \bar{z}_l)\overline{r_{2\kappa-m-k}}$, $k = 0, 1, \ldots, m$.

(2) *If* $2\kappa - m < 0$, *it has only the trivial solution.*

Proof. We first show that $u_0(z)$ is bounded in $\bar{\Omega}$. Let $f(z)$ be defined by

$$f(z) = \begin{cases} \pi(z)u_0(z), & |z| < 1, \\ \overline{\pi\left(\frac{1}{\bar{z}}\right)u_0\left(\frac{1}{\bar{z}}\right)}, & |z| > 1. \end{cases}$$

Then on the boundary we have

$$f^{+}(z) = \pi(z)u_0^{+}(z), \quad f^{-}(z) = \overline{\pi(z)u_0^{+}(z)},$$

where $f^+(z) = \lim_{\substack{z_0 \to z \\ |z_0|<1}} f(z_0)$, $f^-(z) = \lim_{\substack{z_0 \to z \\ |z_0|>1}} f(z_0)$, which exist on $\partial\Omega$ since $\beta_l < 1$. Hence, we have on the boundary $\partial\Omega$

$$\bar{z}^\kappa f^+(z) + z^\kappa f^-(z) = 0,$$

or

$$f^+(z) = -z^{2\kappa} f^-(z) \text{ on } \partial\Omega, \text{ when } \kappa \geqslant 0,$$

$$-z^{-2\kappa} f^+(z) = f^-(z) \text{ on } \partial\Omega, \text{ when } \kappa \leqslant 0.$$

By Liouville's theorem we have respectively

$$f(z) = \begin{cases} p_{2\kappa}(z), & |z|<1, \\ -z^{-2\kappa} p_{2\kappa}, & |z|>1, \end{cases} \tag{11}$$

for $\kappa \geqslant 0$, where $p_{2\kappa}(z)$ is a polynomial in z of degree 2κ, and

$$f(z) = \begin{cases} -z^{2\kappa} c, & |z|<1, \\ c, & |z|>1, \end{cases} \tag{12}$$

when $\kappa < 0$, where c is a complex constant.

From (11) and (12) it follows that $u_0(z)$ is a piecewise rational function. Therefore $u_0(z)$ is continuous in $\bar{\Omega}$.

Consider the piecewise holomorphic function $F(z)$ defined by

$$F(z) = \begin{cases} u_0(z), & |z|<1, \\ \overline{u_0\left(\frac{1}{\bar{z}}\right)}, & |z|>1. \end{cases} \tag{13}$$

Then we have from (10)

$$\bar{z}^\kappa \pi(z) F^+(z) + z^\kappa \overline{\pi(z)} F^-(z) = 0 \quad \text{on } \partial\Omega. \tag{14}$$

From $\bar{z} - \overline{z_l} = (z_l - z)\overline{z z_l}$ for $|z| = |z_l| = 1$ it follows that when $|z| = 1$

$$\overline{\pi(z)} = (-1)^m \bar{z}^m \left(\prod_{l=1}^m \overline{z_l} \right) \pi(z).$$

Since $u_0(z)$ is continuous on $\bar{\Omega}$, by virtue of (14) we have for $|z| = 1$, $z \neq z_l$,

$$\bar{z}^\kappa F^+(z) + (-1)^m \left(\prod_{l=1}^m \overline{z_l} \right) \bar{z}^m z^\kappa F^-(z) = 0 \quad \text{on } \partial\Omega. \tag{15}$$

From (15), Lemma 2 is proved. $\square$

We now consider the in-homogeneous problem

$$\begin{cases} u_{0\bar{z}} = 0 & \text{in } \Omega, \\ \operatorname{Re}[\bar{z}^k \pi(z) u_0(z)] = \tilde{g}(z) & \text{on } \partial\Omega. \end{cases} \tag{16}$$

We have the following necessary condition for its solvability.

Lemma 3. *If problem* (16) *is solvable* (*in the sense of* Problem A) *then*

$$\tilde{g}(z_l) = 0, \quad l = 1, 2, \ldots, m. \tag{17}$$

Proof. Since $u_0(z) = O(|z - z_l|^{-\beta_l})$, $\beta_l < 1$, from (16) we get

$$|\tilde{g}(z)| \leqslant |\bar{z}^\kappa| \prod_{l=1}^{m} |z - z_l| \, |u_0(z)|,$$

which proves Lemma 3. □

Remark 5. For problem (16) (and Problem A), if one is satisfied with solutions with singularities of order greater than or equal to 1, condition (17) does not appear. But the number of solutions changes also. If one requires that $u_0(z)$ has a zero of order greater than zero, the function $\tilde{g}(z)$, together with its derivatives, must satisfy (17) too.

Let condition (17) hold for problem (16). Then from $\tilde{g} \in C^\alpha(\partial\Omega)$ it follows that

$$\tilde{g}(z) = O(|z - z_l|^\alpha), \quad l = 1, 2, \ldots, m, \text{ on } \partial\Omega.$$

We now seek a particular solution of problem (16).

Using (13) from (16) we get

$$\bar{z}^\kappa \pi(z) F^+(z) + z^\kappa \overline{\pi(z)} F^-(z) = 2\tilde{g}(z) \quad \text{on } \partial\Omega.$$

Hence when $2\kappa - m \geqslant 0$ we get

$$F^+(z) = (-1)^{m-1} \left(\prod_{l=1}^{m} \overline{z_l} \right) z^{2\kappa - m} F^-(z) + \frac{2z^\kappa \tilde{g}(z)}{\pi(z)}, \text{ on } \partial\Omega,$$

from which we get a particular solution for $\{F^+, F^-\}$

$$F^+(z) = \frac{1}{2\pi i} \int_{|t|=1} \frac{2t^\kappa \tilde{g}(t)}{\pi(t)(t - z)} dt, \quad |z| < 1,$$

$$F^-(z) = \frac{(-1)^{m-1} \left(\prod_{l=1}^{m} z_l \right) z^{-2\kappa + m}}{2\pi i} \int_{|t|=1} \frac{2t^\kappa \tilde{g}(t)}{\pi(t)(t - z)} dt, \quad |z| > 1.$$

Thus a particular solution is

$$\begin{aligned}\tilde{u}(z) =& \frac{1}{2}\left[F^+(z) + \overline{F^-\left(\frac{1}{\bar{z}}\right)}\right]\\ =& \frac{1}{2\pi i}\int_{|t|=1}\frac{t^\kappa \tilde{g}(t)}{\pi(t)(t-z)}dt + \frac{(-1)^m\left(\prod_{l=1}^m \overline{z_l}\right)z^{2\kappa-m}}{2\pi i}\int_{|t|=1}\frac{\bar{t}^\kappa \tilde{g}(t)}{\overline{\pi(t)}\left(\bar{t}-\frac{1}{z}\right)}d\bar{t}\\ =& \frac{1}{2\pi i}\int_{|t|=1}\frac{t^\kappa \tilde{g}(t)}{\pi(t)(t-z)}dt + \frac{(-1)^m\left(\prod_{l=1}^m \overline{z_l}\right)z^{2\kappa-m+1}}{2\pi i}\int_{|t|=1}\frac{\bar{t}^\kappa \tilde{g}(t)}{\overline{\pi(t)}(t-z)t}dt. \qquad (18)\end{aligned}$$

The above formula is also valid when $2\kappa - m = -1$.

When $2\kappa - m \leqslant -2$ and

$$\int_{|t|=1}\frac{\bar{t}^\kappa \tilde{g}(t)}{\overline{\pi(t)}t^{k+1}}dt = 0, \quad k = 1, 2, \dots, |2\kappa - m| - 1, \qquad (19)$$

the function

$$\tilde{u}(z) = \frac{1}{2\pi i}\int_{|t|=1}\frac{t^\kappa \tilde{g}(t)}{\pi(t)(t-z)}dt + \frac{(-1)^m\prod_{l=1}^m \overline{z_l}}{2\pi i}\int_{|t|=1}\frac{\bar{t}^\kappa \tilde{g}(t)}{\overline{\pi(t)}t^{|2\kappa-m|}(t-z)}dt \qquad (20)$$

satisfies $\tilde{u}(z) = O(|z - z_l|^{\alpha-1})$ and $\mathrm{Re}[\bar{z}^\kappa \pi(z)\tilde{u}(z)] = \tilde{g}(z)$ on $\partial\Omega$.

Summarizing the above we have proved

Theorem 2. *For the in-homogeneous problem* (16), *we have*

(1) *when* $2\kappa - m \geqslant 0$ *and* (16) *is solvable then* (17) *holds. Conversely if* (17) *holds the problem is always solvable and its solutions are given by*

$$u_0(z) = \tilde{u}(z) + \sum_{k=0}^{2\kappa-m} r_k z^k,$$

where $\tilde{u}(z)$ *is defined by* (18), $r_k = (-1)^{m-1}\left(\prod_{l=1}^m \overline{z_l}\right)\overline{r_{2\kappa-m-k}}$ $(k = 0, 1, \dots, 2\kappa - m)$.

(2) *when* $2\kappa - m = -1$ *and* (17) *holds, it has a unique solution given by* (20).

(3) *when* $2\kappa - m \leqslant -2$, (16) *is solvable if* (17) *and* (19) *hold. The unique solution is given by* (20).

Remark 6. In (19), not all of these equations are independent, see (28) below.

3.2. A modified problem

By virtue of Theorem 2, to assure uniqueness of solution, we must add some additional conditions when $2\kappa - m \geqslant 0$. While when $2\kappa - m \leqslant -2$ we need to modify the boundary condition so that the solvability conditions (19) are satisfied automatically. We therefore consider the modified problem corresponding to (16)

$$u_{\bar{z}} = 0, \quad |z| < 1, \tag{21}$$

$$\mathrm{Re}[\bar{z}^{\kappa}\pi(z)u(z)] = g(z) + h(z), \quad |z| = 1. \tag{22}$$

Moreover, if $2\kappa - m \geqslant 0$

$$\mathrm{Im}[\overline{t_n}\pi(t_n)u(t_n)] = b_n, \quad n = 0, 1, \ldots, 2\kappa - m, \tag{23}$$

where $\{t_n : n = 0, 1, \ldots, 2\kappa - m\} \subset \partial\Omega$ are $2\kappa - m + 1$ different points and $\{t_n\} \cap \{z_l\} = \emptyset$, the numbers $\{b_n\}$ are arbitrary constants. The function h is defined by

$$h(z) = \begin{cases} 0 & \text{if } 2\kappa - m \geqslant -1, \\ \overline{\pi(z)} \sum_{j=1+\kappa}^{m-\kappa-1} h_j z^j & \text{if } 2\kappa - m \leqslant -2, \end{cases} \tag{24}$$

where the constants $\{h_j\}$ satisfy

$$h_j = (-1)^m \left(\prod_{l=1}^{m} z_l \right) \overline{h_{m-j}}, \quad j = 1 + \kappa, \ldots, m - \kappa - 1, \tag{25}$$

so that $h(z)$ is a real-valued function on $\partial\Omega$. When $2\kappa - m \leqslant -2$ we need to determine the function u as well as the constants $\{h_j\}$.

Theorem 3. *Problem* (21)–(23) *is uniquely solvable if and only if the function g satisfies*

$$g(z_l) = 0, \quad l = 1, 2, \ldots, m.$$

Proof. From (24), by reasoning in the same way as in the proof of Lemma 3, we get the necessity part of Theorem 3. We now show the sufficiency part.

We first consider the case $2\kappa - m \geqslant 0$. By Theorem 2, the solution of (21) and (22) can be represented as

$$u(z) = \tilde{u}(z) + p(z),$$

where $\tilde{u}(z)$ is defined by (18) and $p(z) = \sum_{k=0}^{2\kappa-m} r_k z^k$ is a polynomial of degree $2\kappa - m$, with

$$r_k = (-1)^{m-1} \left(\prod_{l=1}^{m} \overline{z_l} \right) \overline{r_{2\kappa-m-k}}, \quad k = 0, 1, \ldots, 2\kappa - m. \tag{26}$$

From (23) we get $\mathrm{Im}[\overline{t_n}^{\kappa}\pi(t_n)p(t_n)] = \tilde{b_n}$, where $\tilde{b_n} = b_n - \mathrm{Im}[\overline{t_n}^{\kappa}\pi(t_n)\tilde{u}(t_n)]$.

From $\mathrm{Re}[\overline{t_n}\pi(t_n)p(t_n)] = 0$ it follows that $\overline{t_n}^{\kappa}\pi(t_n)p(t_n) = i\tilde{b_n},\ n = 0, 1, \dots, 2\kappa - m$. Hence we have

$$\sum_{k=0}^{2\kappa-m} r_k t_n^k = \frac{i\tilde{b_n} t_n^{\kappa}}{\pi(t_n)}, \quad n = 0, 1, \dots, 2\kappa - m. \tag{27}$$

The algebraic system of Eq. (27) is uniquely solvable for $(r_0, r_1, \dots, r_{2\kappa-m})^T$. We now show that this solution satisfies condition (26). By conjugating both sides of (27) we get

$$\sum_{k=0}^{2\kappa-m} \overline{r_k} t_n^{-k} = \frac{-i\tilde{b_n} t_n^{-\kappa}}{\overline{\pi(t_n)}} = \frac{-i\tilde{b_n} t_n^{-\kappa+m}}{(-1)^m \prod_{l=1}^m \overline{z_l}\pi(t_n)},$$

or

$$\begin{aligned}\frac{i\tilde{b_n} t_n^{\kappa}}{\pi(t_n)} &= -\sum_{k=0}^{2\kappa-m} \overline{r_k}(-1)^m \left(\prod_{l=1}^m \overline{z_l}\right) \overline{t_n^m} t_n^{2\kappa} t_n^{-k} \\ &= \sum_{k=0}^{2\kappa-m} \overline{r_k}(-1)^{m-1} \left(\prod_{l=1}^m \overline{z_l}\right) t_n^{-m+2\kappa-k} \\ &= \sum_{k=0}^{2\kappa-m} (-1)^{m-1} \left(\prod_{l=1}^m \overline{z_l}\right) \overline{r_{2\kappa-m-k}} t_n^k,\end{aligned}$$

which shows that $\{(-1)^{m-1}(\prod_{l=1}^m \overline{z_l})\overline{r_{2\kappa-m-k}}\}$ is also a solution of (27). On the other hand, the solution of (27) is unique, we therefore get (26).

Next we consider the case $2\kappa - m \leqslant -2$. We need to seek a function u and constants $\{h_l\}$ such that (21) and (25) are satisfied.

Let

$$d_k = \int_{|t|=1} \frac{g(t)}{\overline{\pi(t)} t^k} dt, \quad k = 2 + \kappa, \dots, m - \kappa.$$

By conjugating both sides of the above equations, we get

$$\overline{d_k} = (-1)^{m-1} \left(\prod_{l=1}^m \overline{z_l}\right) d_{m-k+2}, \quad k = 2 + \kappa, \dots, m - \kappa. \tag{28}$$

From Theorem 2, the solvability condition of problem (21), (22) becomes

$$\int_{|t|=1} \frac{g(t) + h(t)}{\overline{\pi(t)} t^k} dt = 0, \quad k = 2 + \kappa, \dots, m - \kappa. \tag{29}$$

Hence we get

$$\sum_{l=1+\kappa}^{m-\kappa-1} h_l \int_{|t|=1} t^{l-k}\, dt = -d_k, \quad k = 2+\kappa, \ldots, m-\kappa, \tag{30}$$

and

$$h_k = -2\pi i d_{k+1}, \quad k = 1+\kappa, \ldots, m-\kappa-1.$$

By virtue of (28), we have

$$\begin{aligned}\overline{h_k} &= 2\pi i \overline{d_{k+1}} \\ &= 2\pi i(-1)^{m-1}\left(\prod_{l=1}^{m} \overline{z_l}\right) d_{m-k+1} \\ &= 2\pi i(-1)^{m-1}\left(\prod_{l=1}^{m} \overline{z_l}\right) \frac{h_{m-k}}{-2\pi i} \\ &= (-1)^{m}\left(\prod_{l=1}^{m} \overline{z_l}\right) h_{m-k},\end{aligned}$$

which proves (25).

From (29), by virtue of Theorem 2, problem (21), (22) is uniquely solvable. The proof of Theorem 3 is completed. □

3.3. *Estimates for the modified Problem A*

To show existence and uniqueness of solutions of the equation

$$u_{\bar z} = Au + B\bar u + C \quad \text{in } \Omega, \tag{31}$$

subject to the modified boundary conditions (22) and (23), in this section we establish an a priori estimate for its solutions.

We first simplify the boundary conditions so that it will suffice to deal only with the homogeneous problem (31), (22) and (23).

Let $\Phi(z)$ be holomorphic in Ω and satisfy the boundary conditions (22) and (23). Then we have from properties of the Cauchy type integral (cf. [1, p. 157–168])

$$\Phi(z) = O(|z - z_l|^{\alpha-1}) \quad (l = 1, 2, \ldots, m). \tag{32}$$

The function $w = u - \Phi$ satisfies the equation

$$w_{\bar z} = Aw + B\bar w + C_0 \quad \text{in } \Omega \tag{33}$$

and the homogeneous boundary condition (22) and (23), where $C_0 = A\Phi + B\bar\Phi + C$.

By virtue of the Hölder inequality, we have

$$C_0 \in L^{p_3}(\Omega) \quad \text{for } p_3 < \frac{1}{\frac{1}{p} + \frac{1-\alpha}{2}}. \tag{34}$$

Later on we need $p_3 > 2$, which can be achieved if $p\alpha > 2$.

For $f \in L^p(\Omega)$, consider the integral operator T defined by

$$(Tf)(z) = -\frac{1}{\pi} \int \int_{\Omega} \left[\frac{f(\zeta)}{\zeta - z} + \frac{(-1)^m \left(\prod_{l=1}^m \overline{z_l}\right) z^{2\kappa - m+1} \overline{f(\zeta)}}{1 - \bar{\zeta} z} \right] d\xi\, d\eta.$$

We have [1,19] for the operator T

Lemma 4. *Let $f \in L^p(\Omega)$, $p > 2$ and $2\kappa - m \geqslant 0$. Then*

(i) $Tf \in W_p^1(\Omega)$, $(Tf)_{\bar{z}} = f$,
(ii) $||Tf||_{C^{(p-2)/p}(\bar{\Omega})} \leqslant K_1 ||f||_p$,
(iii) $\mathrm{Re}[\bar{z}^\kappa \pi(z)(Tf)(z)] = 0$ *on* $\partial\Omega$,

where K_1 is a constant independent of f.

Lemma 5. *Let $p > \max\left(2, \frac{2}{\alpha}\right)$ and w be a solution to problem* (33), (22) *and* (23). *Then w satisfies an estimate of the form*

$$||w||\, W_{p_4}^1(\Omega) \leqslant K_2 ||C_0||_{p_4}, \tag{35}$$

where K_2 is a constant independent of C_0 and $p_3 > 2$ satisfies (34), $2 < p_4 \leqslant p_3$.

Proof. Let w be a solution of problem (33), (22) and (23) and the function v be the unique solution of

$$\begin{cases} v_{\bar{z}} = A + B\dfrac{\bar{w}}{w} & \text{in } \Omega, \\ \mathrm{Im}\, v(z) = 0 & \text{on } \partial\Omega, \\ \mathrm{Re}\, v(1) = 0, \end{cases}$$

then v satisfies [4,6]

$$||v||_{W_{p_4}^1(\Omega)} \leqslant K_3. \tag{36}$$

We now consider three cases according to $2\kappa - m$:

(i) $2\kappa - m \geqslant 0$.

Set

$$u = e^{-v} w - T(C_0 e^{-v}).$$

Then by virtue of Lemma 4, we have

$$\begin{cases} u_{\bar{z}} = 0 & \text{in } \Omega, \\ \operatorname{Re}[\bar{z}^{\kappa}\pi(z)u(z)] = 0 & \text{on } \partial\Omega, \\ \operatorname{Im}[\overline{t_n}^{\kappa}\pi(t_n)u(t_n)] = b_n^0, & n = 0, 1, \dots, 2\kappa - m, \end{cases} \tag{37}$$

where $b_n^0 = -\operatorname{Im}[\overline{t_n}^{\kappa}\pi(t_n)T(C_0e^{-v})(t_n)]$.

From Lemma 4 and (36) we have

$$\begin{aligned} |b_n^0| &= |-\operatorname{Im}[\overline{t_n}^{\kappa}\pi(t_n)T(C_0e^{-v})(t_n)]| \\ &\leqslant 2^m|T(C_0e^{-v})(t_n)| \leqslant 2^m K_1 ||C_0e^{-v}||_{p_4} \\ &\leqslant K_4||C_0||_{p_4}. \end{aligned} \tag{38}$$

By virtue of Theorem 3, problem (37) has a unique solution

$$u(z) = \sum_{k=0}^{2\kappa-m} r_k z^k, \tag{39}$$

where $\{r_k\}$ is the unique solution of the algebraic system of equations

$$\sum_{k=0}^{2\kappa-m} r_k t_n^k = \frac{ib_n^0 t_n^{\kappa}}{\pi(t_n)}, \quad n = 0, 1, \dots, 2\kappa - m, \tag{40}$$

with $r_k = (-1)^{m-1}\left(\prod_{l=1}^m \overline{z_l}\right)\overline{r_{2\kappa-m-k}}$, $k = 0, 1, \dots, 2\kappa - m$.

From (40) and (38), we get

$$|r_k| \leqslant K_5 \sum_{n=0}^{2\kappa-m} |b_n^0| \leqslant K_6||C_0||_{p_4}. \tag{41}$$

Combining (39) and (41), we get

$$||u||_{W^1_{p_4}(\Omega)} \leqslant C_7||C_0||_{p_4}. \tag{42}$$

Hence from (36) and (42), we get for the function $w = (u + T(C_0e^{-v}))e^v$ the estimate (35) when $2\kappa - m \geqslant 0$.

(ii) $2\kappa - m = -1$

The above reasoning is still valid except from (37) we have $u = 0$. Hence $w = e^v T(C_0e^{-v})$.

(iii) $2\kappa - m \leqslant -2$

Set $u = e^{-v}w - T_1(C_0e^{-v})$, where the operator T_1 is defined by

$$(T_1 f)(z) = -\frac{1}{\pi}\int\int_{\Omega} \frac{f(\zeta)}{\zeta - z}\, d\xi\, d\eta.$$

Then we have

$$\begin{cases} u_{\bar{z}} = 0 & \text{in } \Omega, \\ \mathrm{Re}[\bar{z}^{\kappa}\pi(z)u(z)] = e^{-v}h(z) + g^0(z) & \text{on } \partial\Omega, \end{cases} \tag{43}$$

where $g^0(z) = \mathrm{Re}[\bar{z}^{\kappa}\pi(z)(-T_1(C_0 e^{-v}))]$.

From $\pi(z_l) = 0$ it follows that $g^0(z_l) = 0$ $(l = 1, 2, \ldots, m)$. Hence from Theorems 2 and 3, problem (43) is uniquely solvable. Parallel to (29) and (30), we have by virtue of Theorems 2 and 3

$$\begin{aligned} u(z) =& \frac{1}{2\pi i}\int_{|t|=1} \frac{t^{\kappa}(e^{-v}h(t) + g^0(t))}{\pi(t)(t-z)}\,dt \\ &+ \frac{(-1)^m \prod_{l=1}^m \overline{z_l}}{2\pi i}\int_{|t|=1} \frac{\bar{t}^{\kappa}(e^{-v}h(t) + g^0(t))}{\overline{\pi(t)}t^{|2\kappa-m|}(t-z)}\,dt \\ =& \frac{1}{2\pi i}\int_{|t|=1} \frac{t^{\kappa}e^{-v}h(t)}{\pi(t)(t-z)}\,dt \\ &+ \frac{(-1)^m \prod_{l=1}^m \overline{z_l}}{2\pi i}\int_{|t|=1} \frac{\bar{t}^{\kappa}e^{-v}h(t)}{\overline{\pi(t)}t^{|2\kappa-m|}(t-z)}\,dt, \end{aligned} \tag{44}$$

where $h(z) = \pi(z)\sum_{j=1}^{m-\kappa-1} h_j z^j$ and $\{h_j\}$ are determined by the system of algebraic equations

$$\sum_{l=1+\kappa}^{m-\kappa-1} h_l \int_{|t|=1} e^{-v}t^{l-k}\,dt = -d_k^0, \quad k = 2+\kappa, \ldots, m-\kappa, \tag{45}$$

where

$$d_k^0 = \int_{|t|=1} \frac{g^0(t)}{\overline{\pi(t)}t^k}\,dt, \quad k = 2+\kappa, \ldots, m-\kappa.$$

The system of algebraic equations (45) is uniquely solvable with respect to $\{h_l\}$ (cf. [1, p. 57–58]). Moreover, we have by substituting g^0 into d_k^0

$$\begin{aligned} |h_l| \leqslant K_8 \sum_{k=2+\kappa}^{m-\kappa} |d_k^0| &\leqslant K_9 ||T_1(C_0 e^{-v})||_{C^0(\bar{\Omega})} \\ &\leqslant K_{10}||C_0||_{p_4} \quad (l = 1, 2, \ldots, m-\kappa-1). \end{aligned}$$

Hence from (44) we get

$$||u||_{W^1_{p_4}(\Omega)} \leqslant K_{11}||C_0||_{p_4}. \tag{46}$$

From (36) and (46), estimate (35) follows when $2\kappa - m \leqslant -2$. The proof of Lemma 5 is completed. $\square$

3.4. Solution of the Problem A

We are now in a position to show the existence of solutions of problem A. For convenience of presentation, we first work with the modified boundary conditions and then turn to the general case. We have

Theorem 4. *Let $p\alpha > 2$. Then the modified problem* (7) *and* (22), (23) *is solvable if and only if*

$$g(z_l) = 0, \quad l = 1, 2, \dots, m. \tag{47}$$

Proof. From Lemma 3, the 'only if' part follows. We now show that under condition (47), problem (7) and (22), (23) is uniquely solvable.

We introduce the following problem, which depends on a parameter η with $0 \leqslant \eta \leqslant 1$:

$$\begin{cases} u_{\bar{z}} = \eta(Au + B\bar{u}) + C & \text{in } \Omega, \\ \mathrm{Re}[\overline{G(z)}\pi(z)u(z)] = g(z) + h(z) & \text{on } \partial\Omega. \end{cases} \tag{48}$$

Moreover if $2\kappa - m \geqslant 0$,

$$\mathrm{Im}[\overline{G(t_n)}\pi(t_n)u(t_n)] = b_n, \quad n = 0, 1, \dots, 2\kappa - m, \tag{49}$$

where the function h is defined in (24), (25).

If $\eta = 0$, problem (48), in (49) is solved the same way as we did in the proof of Lemma 5, but we need to set now $v = 0$ and $C = C_0$. We show that if problem (48), (49) is solvable for $\eta = \eta_0$ $(0 \leqslant \eta_0 < 1)$, then there exists a δ which is independent of η_0 such that for every η satisfying $\eta_0 < \eta \leqslant \eta_0 + \delta$, problem (48), (49) is still solvable.

We construct a sequence $\{u^n\}$ by taking $u^0 = u(z, \eta_0)$ as the solution of problem (48),(49) when $\eta = \eta_0$, and for $n \in \mathbb{N}$ by the following successive approximation scheme:

$$\begin{cases} u_{\bar{z}}^{n+1} = \eta_0(Au^{n+1} + B\overline{u^n}) + (\eta - \eta_0)(Au^n + Bu^{\bar{n}}) + C & \text{in } \Omega, \\ \mathrm{Re}[\overline{G(z)}\pi(z)u^{n+1}(z)] = g(z) + h(z) & \text{on } \partial\Omega, \\ \mathrm{Im}[\overline{G(t_k)}\pi(t_k)u^{n+1}(t_k)] = b_k, k = 0, 1, \dots, 2\kappa - m, & \text{if } 2\kappa - m \geqslant 0. \end{cases} \tag{50}$$

When $u^n \in W^1_{p_4}(\Omega)$, where p_4 is the same as in Lemma 5, Eq. (50) has coefficients ηA, $\eta B \in L^p(\Omega)$ and the free term $(\eta - \eta_0)(Au^n + B\overline{u^n}) \in L^p(\Omega)$ since $u^n \in C^0(\Omega)$ by virtue of the Sobolev embedding theorem. Thus, problem (50) is solved from the assumption that (48) is solvable for any C when $\eta = \eta_0$.

We now show that the sequence $\{u^n\}$ defined by (50) is convergent in $W^1_{p_4}(\Omega)$ if $\eta - \eta_0$ is small enough. We consider the difference $u^{n+1} - u^n$ for $n \in \mathbb{N}$, which satisfies

$$\begin{cases} (u^{n+1}-u^n)_{\bar z} = \eta_0[A(u^{n+1}-u^n)+B\overline{(u^{n+1}-u^n)}] & \\ \quad +(\eta-\eta_0)[A(u^n-u^{n-1})+B\overline{(u^n-u^{n-1})}] & \text{in } \Omega, \\ \operatorname{Re}[\overline{G(z)}\pi(z)(u^{n+1}(z)-u^n(z))] = h(z) & \text{on } \partial\Omega, \\ \operatorname{Im}[\overline{G(t_k)}\pi(t_k)(u^{n+1}(t_k)-u^n(t_k))] = 0, k=0,1,\dots,2\kappa-m, & \text{if } 2\kappa-m\geqslant 0. \end{cases}$$

By virtue of Lemma 5 and the fact that $\eta_0<1$, we have the estimate

$$\begin{aligned} ||u^{n+1}-u^n||_{W^1_{p_4}(\Omega)} &\leqslant K_2(\eta-\eta_0)||A(u^n-u^{n-1})+B\overline{(u^n-u^{n-1})}||_{p_4} \\ &\leqslant K_{12}(\eta-\eta_0)||u^n-u^{n-1}||_{W^1_{p_4}(\Omega)}. \end{aligned}$$

Let $\delta=\frac{1}{2K_{12}}$. Then for every η satisfying $\eta_0<\eta\leqslant\eta_0+\delta$, we have

$$\begin{aligned} ||u^{n+1}-u^n||_{W^1_{p_4}(\Omega)} &\leqslant \frac{1}{2}||u^n-u^{n-1}||_{W^1_{p_4}(\Omega)} \\ &\leqslant \left(\frac{1}{2}\right)^n ||u^1-u^0||_{W^1_{p_4}(\Omega)}. \end{aligned}$$

This shows that the sequence $\{u^n\}$ is convergent in $W^1_{p_4}(\Omega)$, and the limit function u is a solution to (48), (49). Since δ is independent of η_0, we can repeat the above procedure finitely many times until we arrive at $\eta=1$. This completes the proof of Theorem 4. □

Returning to Problem A, we have

Theorem 5. *If Problem A is solvable then condition* (47) *holds. When* (47) *holds and*

(1) $2\kappa-m\geqslant 0$, *the homogeneous problem has exactly* $2\kappa-m+1$ *linearly independent solutions over the field of real numbers*, *the inhomogeneous problem is always solvable*.

(2) $2\kappa-m=-1$, *it has a unique solution*.

(3) $2\kappa-m\leqslant -2$, *the homogeneous problem has only the trivial solution*, *Problem A is solvable if and only if the functions C and g satisfy* $m-2\kappa-1$ *real solvability conditions*.

For the regularity of solutions of the Problem A, from (32) we have

Lemma 6. *Under the assumption of Theorem* 4, *let u be a solution of Problem A*, *then* $u\in C^0(\bar\Omega\backslash\bigcup_{l=1}^m\{z_l\})\cap W^1_{p_4}(\Omega_\varepsilon)$ *and* $u(z)=O(|z-z_l|^{\alpha-1})$ $(l=1,2,\dots,m)$.

4. Problem RH when $|z_l| = 1$ $(l = 1, 2, \dots, m)$

In this section, we assume that all the zeros of the polynomial $P(z)$ are on the boundary $\partial\Omega$, i.e., $|z| = 1$, $l = 1, 2, \dots, m$.

We restrict ourselves to the case $\operatorname{Im} a_l = 0$ $(l = 1, 2, \dots, m)$.

For $l = 1, 2, \dots, m$, we let

$$w_l(z) = \begin{cases} \dfrac{(\bar{z} - \overline{z_l})^{n_l}}{(z - z_l)^{n_l - 1}}, & \lambda_l = 0, \\ \dfrac{(\bar{z} - \overline{z_l})^{n_l} |z - z_l|^{2\lambda_l}}{(z - z_l)^{n_l}}, & 0 < \lambda_l \leqslant \dfrac{1}{2}, \\ \dfrac{(\bar{z} - \overline{z_l})^{n_l + 1} |z - z_l|^{2(\lambda_l - 1)}}{(z - z_l)^{n_l}}, & \dfrac{1}{2} < \lambda_l < 1. \end{cases} \tag{51}$$

We consider only three cases:

(i) $\lambda_l = 0$ $(l = 1, 2, \dots, m)$,
(ii) $0 < \lambda_l \leqslant \frac{1}{2}$ $(l = 1, 2, \dots, m)$,
(iii) $\frac{1}{2} \leqslant \lambda_l < 1$ $(l = 1, 2, \dots, m)$.

With the transform

$$w(z) = \left(\prod_{l=1}^{m} w_l(z) \right) \varphi(z), \tag{52}$$

Problem RH becomes

$$\begin{cases} \varphi_{\bar{z}} = a\varphi + b_1 \bar{\varphi}, & |z| < 1, \\ \operatorname{Re}[\overline{G(z)} \prod_{l=1}^{m} w_l(z) \varphi(z)] = g(z), & |z| = 1, \end{cases} \tag{53}$$

where $b_1(z) = b(z) \overline{\prod_{l=1}^{m} w_l(z)} / \prod_{l=1}^{m} w_l(z) \in L^p(\Omega)$.

Remark 7. If $\operatorname{Im} a_l \neq 0$ there will be a factor $\exp(2i \operatorname{Im} a_l \ln|z - z_l|)$ in the definition of $w_l(z)$ in (51). Since $z_l \in \partial\Omega$ $\ln|z - z_l|$ will be unbounded when $z \in \partial\Omega$. Then we are lead to a Riemann–Hilbert problem with infinite index. We do not discuss this here. For the Riemann problem with infinite index, one is referred to [9,11,12].

Since we are seeking continuous solutions of Problem RH, through transform (52), the function φ can have a singularity at z_l $(l = 1, 2, \dots, m)$, respectively, $\varphi(z) =$

$O(|z-z_l|^{-\beta_l})$, where β_l satisfies

$$\begin{aligned}
&\beta_l<1 && \text{if } \lambda_l=0,\ l=1,2,\dots,m,\\
&\beta_l<2\lambda_l && \text{if } 0<\lambda_l\leqslant\frac{1}{2},\ l=1,2,\dots,m,\\
&\beta_l<2\lambda_l-1 && \text{if } \frac{1}{2}<\lambda_l<1,\ l=1,2,\dots,m.
\end{aligned}$$

From (53), we have

Lemma 7. *If problem* (53) *is solvable then*

$$g(z_l)=0,\quad l=1,2,\dots,m. \tag{54}$$

We now consider the three cases.

Case (i): $\lambda_l=0,\ l=1,2,\dots,m$

From (53) we get

$$\begin{cases}\varphi_{\bar z}=a\varphi+b_1\bar\varphi, & |z|<1,\\ \operatorname{Re}[\overline{G_1(z)}\pi(z)\varphi(z)]=g(z), & |z|=1,\end{cases} \tag{55}$$

where $G_1(z)=(-1)^{\sum_{l=1}^m n_l}\left(\prod_{l=1}^m \overline{z_l}^{n_l}\right)z^{\sum_{l=1}^m n_l}G(z)$.

Since $G_1(z)\neq 0$ on $\partial\Omega$, the index of problem (55) is given by

$$\kappa_1=\kappa+\sum_{l=1}^m n_l.$$

From Theorem 5 and Lemma 6 we get

Lemma 8. *Let condition* (54) *hold. Then we have*

(1) *When* $2\kappa_1-m\geqslant 0$, *the homogeneous problem* (55) *has exactly* $2\kappa_1-m+1$ *linearly independent solutions over the field of real numbers*, *the inhomogeneous problem is always solvable*.

(2) *When* $2\kappa_1-m=-1$, *it has a unique solution*.

(3) *When* $2\kappa_1-m\leqslant -2$, *the homogeneous problem has only the trivial solution*; *problem* (55) *is solvable if and only if the function* g *satisfies* $m-2\kappa_1-1$ *real solvability conditions*.

Moreover, *for the solution we have*

$$\varphi(z)=O(|z-z_l|^{\alpha-1}),\quad l=1,2,\dots,m.$$

Case (ii): $0<\lambda_l\leqslant\frac{1}{2},\ l=1,2,\dots,m$

We have from (53) the normal Riemann–Hilbert problem

$$\begin{cases}\varphi_{\bar z}=a\varphi+b_1\bar\varphi, & |z|<1,\\ \operatorname{Re}[\overline{G_2(z)}\varphi(z)]=g_2(z), & |z|=1,\end{cases} \tag{56}$$

where $G_2(z) = (-1)^{\sum_{l=1}^m n_l}(\prod_{l=1}^m z_l^{n_l})z^{\sum_{l=1}^m n_l}G(z),\ g_2(z) = \frac{g(z)}{\prod_{l=1}^m |z-z_l|^{2\lambda_l}}$. The index of problem (56) is

$$\kappa_2 = \kappa + \sum_{l=1}^m n_l.$$

From properties of Cauchy type integrals, we have

Lemma 9. *Let condition* (54) *hold. Then we have*

(1) *When* $\kappa_2 \geqslant 0$, *the homogeneous problem* (56) *has exactly* $2\kappa_2 + 1$ *linearly independent solutions over the field of real numbers*, *the inhomogeneous problem is always solvable*.
(2) *When* $\kappa_2 < 0$, *the homogeneous problem* (56) *has only the trivial solution*: *the inhomogeneous problem is solvable if and only if the function g satisfies* $1 - 2\kappa_2$ *solvability conditions*.

Moreover, *for the solution we have when* $\alpha - 2\lambda_l < 0$

$$\varphi(z) = O(|z - z_l|^{\alpha - 2\lambda_l}), \quad l = 1, 2, \dots, m,$$

otherwise φ *is continuous at* z_l.

Case (iii): $\frac{1}{2} < \lambda_l < 1,\ l = 1, 2, \dots, m$
From (53) we get

$$\begin{cases} \varphi_{\bar z} = a\varphi + b_1\bar\varphi, & |z| < 1, \\ \mathrm{Re}[\overline{G_3(z)}\pi(z)\varphi(z)] = g_3(z), & |z| = 1, \end{cases} \tag{57}$$

where

$$G_3(z) = (-1)^{\sum_{l=1}^m n_l + m}\left(\prod_{l=1}^m z_l^{n_l+1}\right)z^{\sum_{l=1}^m n_l + m}G(z), \quad g_3(z) = \frac{g(z)}{\prod_{l=1}^m |z - z_l|^{2\lambda_l - 1}}.$$

The index of problem (57) is

$$\kappa_3 = \kappa + \sum_{l=1}^m n_l + m.$$

From Theorem 5 and Lemma 6 we have

Lemma 10. *Let condition* (54) *hold. Then we have*
(1) *When* $2\kappa_3 - m \geqslant 0$, *the homogeneous problem* (57) *has exactly* $2\kappa_3 - m + 1$ *linearly independent solutions over the field of real numbers*, *the non-homogeneous problem is always solvable*.
(2) *When* $2\kappa_3 - m = -1$, *it has a unique solution*.

(3) *When* $2\kappa_3 - m \leqslant -2$, *the homogeneous problem has only the trivial solution; problem* (55) *is solvable if and only if the function* g *satisfies* $m - 2\kappa_3 - 1$ *real solvability conditions.*

Moreover, for the solution we have when $\lambda_l > \frac{1+\alpha}{2}$

$$\varphi(z) = O(|z - z_l|^{l+\alpha-2\lambda_l}), \quad l = 1, 2, \ldots, m,$$

otherwise φ *is continuous at* z_l.

For Problem RH, from Lemmas 7–10 and (52) we finally get

Theorem 6. *Problem RH is solvable only if condition* (54) *holds. Conversely if* (54) *is satisfied then we have the following*:

If $2\kappa + \sum_{l=1}^{m} \sigma_l \geqslant 0$, *the homogeneous problem has exactly* $2\kappa + \sum_{l=1}^{m} \sigma_l + 1$ *linearly independent solutions over the field of real numbers*, *the inhomogeneous problem is always solvable*; *if* $2\kappa + \sum_{l=1}^{m} \sigma_l = -1$, *it has a unique solution*; *when* $2\kappa + \sum_{l=1}^{m} \sigma_l \leqslant -2$, *the homogeneous problem has only the trivial solution*, *the homogeneous problem is solvable if and only if the function* g *satisfies* $\left|2\kappa + \sum_{l=1}^{m} \sigma_l\right| - 1$ *real solvability conditions*.

References

[1] H. Begehr, Complex Analytic Methods for Partial Differential Equations. An Introductory Text, World Scientific, Singapore, 1994.

[2] H. Begehr, R.P. Gilbert, On Riemann boundary value problem for certain linear elliptic systems in the plane, J. Differential Equations 32 (1979) 1–14.

[3] H. Begehr, G.N. Hile, Nonlinear Riemann boundary value problem for a semilinear elliptic system in the plane, Math. Z. 179 (1982) 241–261.

[4] H. Begehr, G.C. Hsiao, The Hilbert boundary value problem for nonlinear elliptic systems, Proc. Roy. Soc. Edinburgh 94A (1983) 97–112.

[5] L. Bers, Theory of Pseudo-analytic Functions, Lecture Notes, New York Univ., Inst. Math. Mech., New York, 1953.

[6] D.Q. Dai, Piecewise continuous solutions of nonlinear elliptic equations in the plane, Acta Math. Sci. Ser. B 14 (1994) 383–392.

[7] D.Q. Dai, On some problems for first order elliptic systems in the plane, in: H. Begehr, O. Celebi, W. Tutschke (Eds.), Complex Methods for Partial Differential Equations, Kluwer Academic Publishers, Dordrecht, 1999, pp. 21–27.

[8] D.Q. Dai, W. Lin, Piecewise continuous solutions of nonlinear pseudoparabolic equations in two space dimensions, Proc. Roy. Soc. Edinburgh 121A (1992) 203–217.

[9] W. Haack, W. Wendland, Partial and Pfaffian Differential Equations, Pergamon Press, Oxford, 1972.

[10] L.G. Mikhailov, A New Class of Singular Equations and its Application to Differential Equations with Singular Coefficients, Akademie-Verlag, Berlin, 1970.

[11] V.N. Monakhov, E.V. Semenko, Well-posedness classes of junction boundary value problems for analytic functions with infinite index, Soviet Math. Dokl. 33 (1986) 17–20.

[12] I.V. Ostrovskii, Conditions for solvability of homogeneous Riemann boundary value problems with infinite index, J. Soviet Math. 59 (1992) 583–598.

[13] M. Reissig, A. Timofeev, Special Vekua equations with singular coefficients, Appl. Anal. 73 (1999) 187–199.

[14] A. Tungatarov, On the theory of Carleman–Vekua equation with a singular point, Sbornik Math. 78 (1994) 357–365.
[15] A. Tungatarov, Continuous solutions of the generalized Cauchy–Riemann system with a finite number of singular points, Mat. Zametki 56 (1994) 105–115 (in Russian).
[16] Z.D. Usmanov, Generalized Cauchy–Riemann Systems with a Singular Point, Longman, Harlow, 1997.
[17] I.N. Vekua, Generalized Analytic Functions, Pergamon, Oxford, 1962.
[18] I.N. Vekua, On one class of elliptic systems with singularities, Proc. Conf. Funct., Tokyo 1969 (1970), pp. 142–147.
[19] G.C. Wen, H. Begehr, Boundary Value Problems for Elliptic Equations and Systems, Longman, Harlow, 1990.